SPECIAL KINETIC MODELS OF SELECTION PROCESSES IN BIOMACROMOLECULAR SYSTEMS

Vladimír KVASNIČKA

Department of Mathematics,
Slovak Institute of Technology, 812 37 Bratislava

Received January 5th, 1988
Accepted February 28th, 1988

Physico-chemical models of selection processes on biomacromolecules — replicators are investigated. Three special cases are considered: *1*) Replicators are selfreproduced with mutations, *2*) selection process involving two or more substrates, and *3*) selection process is controlled by external positive inflows of replicators. Simple tools of qualitative theory of differential equations are used, in particular the so-called linearization method based on the eigenvalues of Jacobi matrix evaluated at the given stationary state.

A mathematical modelling of selection processes on molecular level was initiated by Eigen[1-2] (cf. also refs[3-4]) at the beginning of seventies. He introduced the notion *information carrier*, which corresponds to a biomacromolecular system capable to reproduce itself. This conceptual notion in the forthcoming part of present communication will be called the *replicator*[5]. Eigen has described the process of self-reproduction of replicators by phenomenological differential equations with two types of external constraints which would make the reaction system more competitive. The first type of these constraints is called the *constant population*, it requires that the sum of concentrations of replicators is kept fixed during the whole time evolution of the kinetic system. The *second* type, called the *constant fluxes*, requires that an inflow of energy rich molecules (called the substrate) in the reaction system is time-invariant. Both these types of external constraints induce a selective pressure among replicators which gives to rise to typical Darwinian selection known up to that time only for living systems. Eigen has used this phenomenological model as a conceptual tool to abridge a deep gap between information contents of non-instructed synthesis of chemical macromolecules (chemical evolution) and instructed synthesis of biological macromolecules (a very beginning of biological evolution). In the literature[6-8] was mainly studied the selection model involving the constant-population constraint while its counterparts based on the constant-fluxes constraints were considered only marginally[1,8] as a possible and alternative explanation of selection processes on molecular level. This was caused mainly due to mathematical difficulties in the stability analysis of its stationary states. Ebeling et al.[9-10] have shown that the mentioned formal "drawback" of constant-fluxes approach can be simply surmounted by making use of the standard technique of qualitative theory of differential equations[11], in particular by the so-called linearization method based on the negativeness of real parts of eigenvalues of the Jacobi matrix evaluated at the given stationary state. Moreover, the constant-fluxes approach has very easy and simple physico-chemical interpretation, it can be naturally related to known kinetic and ecological models that are forming the main field of interest for up-to-date formal chemical kinetics[12]. Recently, we have enlarged[13] the

Ebeling's theory by taking into account very important possibility that not only replicators but also the substrate are spontaneously decaying. Such an additional requirement removes a "catastrophic" state, overlooked in the Ebeling's approach, in which all replicators are becoming extinct and the substrate concentration is linearly increasing to infinity as $t \to \infty$. We have demonstrated that this unjustly omitted theory represents a very fruitful theoretical possibility how to explain the selection processes on molecular level. Its theoretical tools are very flexible for futher generalization and modification, i.e. it permits to develop very deep and exhaustive particular theoretical studies of the selection processes.

The purpose of this communication is to develop the Ebeling's approach toward the following three particular cases: *1*) The replicator system with erroneous self-reproductions (mutations); *2*) the selection process involving two or more substrates; *3*) the selection process controlled by external constant and non-negative replicator inflows to reaction system. In all these particulat selection models we have introduced the above mentioned possibility of the substrate decay.

THEORETICAL

Section Model with Mutations

We are given a set composed of n replicators (biomacromolecules) $X_1, X_2, \ldots, X_n$ and a substrate X_0, confined to a well stirred reactor, which are capable of replication. It is assumed that each replicator X_i is self-reproduced (replicated) correctly (on itself) or incorrectly (on another replicator) with a participation of the substrate X_0 and that the reactor walls are permeable to energy rich compounds (corresponding to the substrate X_0 with constant inflow) and energy deficient compounds (which are allowed to flow out from the reactor). These assumptions are formally represented by the following system of chemical reactions

$$R_1: \ \square \xrightarrow{\varphi_0} X_0 \,, \tag{A}$$

$$R_2: \ X_0 \xrightarrow{x_0 \psi_0} \square \,, \tag{B}$$

$$A_{ij}: \ X_0 + X_i \xrightarrow{a_{ij} x_0 x_i} X_i + X_j \,, \tag{C}$$

$$B_i: \ X_i \xrightarrow{b_i x_i} \square \,, \tag{D}$$

for $i, j = 1, 2, \ldots, n$. The square symbol[1,2] $\square$ on the r.h.s. of (B) and (D) represents those compounds – reaction products ("garbage") that are irrelevant for the kinetics of studied system. The same symbol was also used on the l.h.s. of (A), here it means a blank side of chemical reaction. Each arrow (reaction) in (A)–(D) is evaluated by the corresponding rate function. The entries $x_0, x_1, x_2, \ldots, x_n$ are the concentra-

tions of $X_0, X_1, X_2, \ldots, X_n$, respectively. The reaction R_1, see (A), is a constant inflow of the substrate X_0. The reaction A_{ij}, see (C), for fixed indices $1 \leqq i, j \leqq n$, represents a replication process of a replicator X_i, this replication is called correct if $i = j$ (i.e. the replicator X_i is reduplicated on itself) or incorrect (then it is called the mutuation) if $i \neq j$ (i.e. the replicator X_i is reduplicated on itself and another replicator X_j). The second reaction R_2, see (B), and the last reactions $B_1, B_2, \ldots, B_n$, see (D), correspond to a decomposition of the substrate and replicators, respectively, to products that are not appearing on the l.h.s. of reactions (A) and (C). Here is naturally to assume that the above mentioned incorrect reduplications – mutations are very rare, or in other words, the non-diagonal rate constants are smaller about few orders than their diagonal counterparts. This means that the rate-constant matrix $A = (a_{ij})$ is classified[14] as a non-negative matrix with dominant diagonal elements (the fact of which will be very important in our forthcoming considerations). In order to keep our theory simplest as possible we shall require that all decomposition rate constants b_i's are the same,

$$b_1 = b_2 = \ldots = b_n = \beta \,. \tag{1}$$

From the standpoint of chemisty this constraint is quite plausible since the decomposition of replicators is much more substantially determined by the physicochemical parameters of the medium in which the replicators are reduplicating than their actual chemical primary structure.

The system $(A)-(D)$ of chemical reactions can be diagrammatically represented by the so-called *reaction graph*[15] (see Fig. 2 in ref.[13]). Following very fruitful idea of Volpert[15,16], the dynamics of considered chemical reactions is described by the following system of differential equations determined over the reaction graph

$$\dot{x}_0 = \varphi_0 - x_0\Big(\sum_{i,j=1}^{n} a_{ij}x_j + \psi_0\Big) \,, \tag{2a}$$

$$\dot{x}_i = x_0 \sum_{j=1}^{n} a_{ij}x_j - \beta x_i \quad (i = 1, 2, \ldots, n) \,. \tag{2b}$$

These differential equations correspond to the fact that the kinetics of $(A)-(D)$ is governed by the *mass-action law*. We have to emphasize, the present model is a generalization of the original Ebeling's approach, we have introduced important assumption that the substrate X_0 is monomolecularly decaying, see (B). It removes a "catastrophic" possiblity in which all replicators $X_1, X_2, \ldots, X_n$ are becoming extinct whereas the substrate concentration is linearly increasing to infinity as $t \to \infty$. If the initial concentrations (at $t = 0$) are positive, then for each $t \geqq 0$ the substrate concentration is positive and replicator concentration are non-negative, and all bounded from above,

$$0 < x_0(t) < \infty\,, \tag{3a}$$

$$0 \leqq x_i(t) < \infty\,. \tag{3b}$$

The matrix form of Eq. (2) is

$$\dot{x}_0 = \varphi_0 - x_0(\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} + \psi_0)\,, \tag{4a}$$

$$\dot{\boldsymbol{x}} = (x_0\boldsymbol{A} - \beta\boldsymbol{E})\,\boldsymbol{x}\,, \tag{4b}$$

where $\boldsymbol{A} = (a_{ij})$ is a square matrix with entries corresponding to rate constants a_{ij}. The symbol $\boldsymbol{E}$ is the unit matrix, $\boldsymbol{e} = (1, 1, \ldots, 1)^{\mathrm{T}}$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ are column vectors.

The eigenproblem of $\boldsymbol{A}$ is

$$\boldsymbol{A}\boldsymbol{c}_i = \lambda_i\boldsymbol{c}_i\,, \quad (i = 1, 2, \ldots, n) \tag{5}$$

where λ_i and $\boldsymbol{c}_i$ are the eigenvalue and the corresponding eigenvector, respectively. Since the matrix $\boldsymbol{A}$ has dominant diagonal, its eigenvalues are roughly equal to the diagonal elements, $\lambda_i \doteq a_{ii}$; the fact of which implies that all eigenvalues are non-degenrate, i.e. none two of them are equal. The matrix $\boldsymbol{U} = (\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n)$ composed of eigenvectors of $\boldsymbol{A}$ is non-singular (since the eigenvectors are linearly independent due to the fact that all eigenvalues are different) and diagonalizes the matrix $\boldsymbol{A}$,

$$\boldsymbol{U}^{-1}\boldsymbol{A}\boldsymbol{U} = \boldsymbol{\Lambda} = \mathrm{dg}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)\,. \tag{6}$$

According to Perron–Frobenius theorem[14] the eigenvector assigned to the most positive eigenvalue is non-negative, i.e. all its entries are non-negative.

Employing the matrix $\boldsymbol{U}$ the differential equations (4*a*, *b*) can be rewritten in the so-called canonical form

$$\dot{t}_0 = \varphi_0 - t_0(\boldsymbol{e}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{t} + \psi_0)\,, \tag{7a}$$

$$\dot{\boldsymbol{t}} = (t_{\mathrm{f}}\boldsymbol{\Lambda} - \beta\boldsymbol{E})\,\boldsymbol{t}\,, \tag{7b}$$

where $t_0 = x_0$ and $\boldsymbol{t} = \boldsymbol{U}^{-1}\boldsymbol{x}$ are new canonical dynamical variables. Their importance consists in a formal similarity with the differential equations of the standard Ebeling's model[10].

Alternative (or second) canonical form of (4*a*, *b*) is (cf. ref.[13])

$$\dot{y}_0 = -y_0^2(\varphi_0 y_0 - \boldsymbol{e}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{y} - \psi_0)\,, \tag{8a}$$

$$\dot{\boldsymbol{y}} = (\boldsymbol{\Lambda} - \beta y_0\boldsymbol{E})\,\boldsymbol{y}\,, \tag{8b}$$

where $y_0 = 1/t_0$, $\boldsymbol{y} = \boldsymbol{t} = \boldsymbol{U}^{-1}\boldsymbol{x}$. This canonical form will be useful to derive the sufficient conditions for a stationary state to be asymptotically stable.

The stationary states of (*7a*, *b*) are of the following two kinds:

1st kind. The stationary concentration $\bar{\imath}_0$ is positive whereas the stationary concentration vector $\bar{\boldsymbol{\imath}}$ is vanishing,

$$\bar{\imath}_0 = \frac{\varphi_0}{\psi_0}, \tag{9a}$$

$$\bar{\boldsymbol{\imath}} = \boldsymbol{0}. \tag{9b}$$

This stationary state will be denoted by S_0.

2nd kind. The stationary concentration $\bar{\imath}_0$ is positive whereas the stationary vector $\boldsymbol{t}$ has all entries vanishing except of a preselected p-th positive entry,

$$t_0 = \frac{\beta}{\lambda_p}, \tag{10a}$$

$$\bar{\boldsymbol{\imath}} = \alpha \boldsymbol{e}_p, \tag{10b}$$

where the positive constant α is determined from the stationarity of (*7a*), i.e. $\varphi_0 = \bar{\imath}_0(\boldsymbol{e}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda}\bar{\boldsymbol{\imath}} + \psi_0)$, we get

$$\alpha = \frac{\lambda_p\varphi_0 - \beta\psi_0}{\beta\lambda_p\boldsymbol{e}^{\mathrm{T}}\boldsymbol{c}_p}. \tag{11}$$

Its positiveness implies

$$\beta\frac{\psi_0}{\varphi_0} < \lambda_p, \tag{12}$$

where we have assumed that $\boldsymbol{c}_p$ is a non-negative vector, $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{c}_p > 0$. This stationary state will be denoted by S_p.

The Jacobi matrix[11] of (*7a*, *b*) is

$$\boldsymbol{J} = \frac{\partial(\dot{\imath}_0, \dot{\boldsymbol{t}})}{\partial(t_0, \boldsymbol{t})} = \begin{pmatrix} -\boldsymbol{e}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{t} - \psi_0 & -t_0\boldsymbol{e}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}\boldsymbol{t} & t_0\boldsymbol{\Lambda} - \beta\boldsymbol{E} \end{pmatrix}. \tag{13}$$

Its specification for the stationary states S_0 and S_p are

$$\boldsymbol{J}(S_0) = \begin{pmatrix} -\psi_0 & -\dfrac{\varphi_0}{\psi_0}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda} \\ 0 & \dfrac{\varphi_0}{\psi_0}\boldsymbol{\Lambda} - \beta\boldsymbol{E} \end{pmatrix}, \tag{14a}$$

$$\boldsymbol{J}(S_p) = \begin{pmatrix} -\frac{\lambda_p}{\beta}\varphi_0 & -\frac{\beta}{\lambda_p}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Lambda} \\ \alpha\lambda_p\boldsymbol{e}_p & \frac{\beta}{\lambda_p}\boldsymbol{\Lambda} - \beta\boldsymbol{E} \end{pmatrix}. \tag{14b}$$

The eigenvalues pf $\boldsymbol{J}(S_0)$ are equal to its diagonal entries (it is a triangular matrix)

$$\mu_0^{(0)} = -\psi_0\,, \tag{15a}$$

$$\mu_i^{(0)} = \frac{\varphi_0}{\psi_0}\lambda_i - \beta\,, \quad (i = 1, 2, \ldots, n)\,. \tag{15b}$$

The eigenvalue $\mu_0^{(0)}$ is automatically negative (the substrate decay rate constant ψ_0 is positive), other eigenvalues $\mu_1^{(0)}, \ldots, \mu_n^{(0)}$ are negative if

$$\beta\frac{\psi_0}{\varphi_0} > \lambda_p = \max_{1 \leq i \leq n} \lambda_i\,. \tag{16}$$

Hence, the stationary state S_0 is asymptotically stable if the above condition (*16*) is fulfilled.

The eigenvalues of $\boldsymbol{J}(S_p)$ are determined by the secular equation $|\boldsymbol{J}(S_p) - \mu\boldsymbol{E}| = 0$,

$$\left(\mu^2 + \frac{\lambda_p}{\beta}\mu + \lambda_p\varphi_0 - \psi_0\beta\right)\prod_{\substack{(i=1)\\ i\neq p}}^{n}\left(\frac{\beta}{\lambda_p}\lambda_i - \beta - \mu\right) = 0\,. \tag{17}$$

The second term on the l.h.s. of (*17*) determines $(n - 1)$ eigenvalues of $\boldsymbol{J}(S_p)$,

$$\mu_i^{(p)} = \frac{\beta}{\lambda_p}\lambda_i - \beta\,, \quad (i = 1, \ldots, p-1, p+1, \ldots, n) \tag{18}$$

The remaining two eigenvalues $\mu_0^{(p)}$ and $\mu_p^{(p)}$ are determined by the quadratic equation (the first term on l.h.s. of (*17*). Its roots (eigenvalues) have negative real parts if and only if the coefficients of the quadratic equation are positive,

$$\beta\frac{\psi_0}{\varphi_0} < \lambda_p\,. \tag{19}$$

The eigenvalues (*18*) are negative if

$$\lambda_p = \max_{1 \leq i \leq n} \lambda_i\,. \tag{20}$$

All eigenvalues of $\boldsymbol{J}(S_p)$ have negative real parts if the conditions (*19*) and (*20*) are simultaneously satisfied

$$\beta \frac{\psi_0}{\varphi_0} < \lambda_p = \max_{1 \leq i \leq n} \lambda_i . \tag{21}$$

This means that among stationary states $S_0, S_1, \ldots, S_n$ is asymptotically stable either S_0. if the condition (*16*) is satisfied, or a state S_p $(1 \leqq p \leqq n)$, if the condition (*21*) is satisfied; therefore the only stationary state is asymptotically stable due to the fact that Eqs (*16*) and (*21*) are mutually excluding. The concentrations of winning stationary state are determined by Eqs (*10a, b*)

$$\bar{x}_0 = \frac{\beta}{\lambda_p} , \tag{22a}$$

$$\bar{\boldsymbol{x}} = \alpha \boldsymbol{U} \boldsymbol{e}_p = \alpha \boldsymbol{c}_p , \tag{22b}$$

where the positive constant α is given by Eq. (*11*). An existence of the non-negative eigenvector $\boldsymbol{c}_p$ assigned to the most positive eigenvalue λ_p is ensured by already mentioned Perron–Frobenius theorem.

The above results represents only necessary conditions for a stationary state to be asymptotically stable. Now we demonstrate that the conditions (*16*) and (*21*) are also sufficient. The equations (*8a, b*) give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{y_0}\right) = \varphi_0 y_0 - \sum_{i=1}^{n} A_i y_i - \psi_0 , \tag{23a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln y_i = \lambda_i - \beta y_0 , \quad (i = 1, 2, \ldots, n) \tag{23b}$$

where

$$A_i = \lambda_i \sum_{j=1}^{n} u_{ji} = \lambda_i \boldsymbol{e}^{\mathrm{T}} \boldsymbol{c}_i , \quad (i = 1, 2, \ldots, n) \tag{23c}$$

Time average concentrations are defined by

$$z_i(t) = \frac{1}{t} \int_0^t y_i(\tau)\, \mathrm{d}\tau , \quad (i = 0, 1, \ldots, n) \tag{24}$$

Let us integrate differential equations (*23a, b*) from $\tau = 0$ to $\tau = t$, after dividing by t, we have

$$\frac{1}{t}\left(\frac{1}{y_0(t)} - \frac{1}{y_0(0)}\right) = \varphi_0 z_0(t) - \sum_{i=1}^{n} A_i z_i(t) - \psi_0 , \tag{25a}$$

$$\frac{\ln y_i(t) - \ln y_i(0)}{t} = \lambda_i - z_0(t)\,\beta\,, \quad (i = 1, 2, \ldots, n) \tag{25b}$$

We shall assume that the limit values $z_i(\infty)$ always exist and they are non-negative and bounded from above. Let us assume that the concentration $y_p = t_p$ does not vanish as $t \to \infty$, fro Eq. (*25b*) we get

$$z_0(\infty) = \frac{\lambda_p}{\beta}\,, \tag{26}$$

this means that such an index p of non-decaying replicator is unambiguously determined. If we have two such indices p and q, then $\lambda_p = \lambda_q$, the condition of which is physically highly unprobable (it implies that the replicators X_p and X_q are kinetically equivalent). For other indices $i = 1, \ldots, p - 1, p + 1, \ldots, n$ the replicator concentrations vanish as $t \to \infty$; limit values of their time averages are vanishing as $t \to \infty$. Since the variable $y_0(t)$ is positive and bounded from above, the relation (*25a*) provides as $t \to \infty$,

$$z_p(\infty) = \frac{\varphi_0}{\beta}\frac{\lambda_p}{A_p} - \frac{\psi_0}{A_p}\,. \tag{27}$$

The limit values of time averages tend to their stationary values $t \to \infty$,

$$\lim_{t\to\infty} z_i(t) = \bar{y}_i\,, \quad (i = 0, 1, \ldots, n) \tag{28}$$

The relation (*25b*) can be simply rewritten in the following form

$$y_i(t) = y_i(0)\exp\{[\lambda_i - z_0(t)\,\beta]\,t\}\,. \tag{29}$$

We shall study asymptotical properties of this solution. For each index $i = 1\ 2\ \ldots$ $\ldots, p - 1, p + 1, \ldots, n$ the concentration vanishes as $t \to \infty$, this implies that the asymptotical value of exponent in Eq. (*29*) should be negative, $\lambda_i - z_0(\infty)\,\beta < 0$, or by making use of the relation (*26*) we get $\lambda_i < \lambda_p$, which is simultaneously with the positiveness of Eq. (*27*) (cf. Eq. (*19*)) equivalent with the condition (*21*) for asymptotical stability of S_p.

Summarizing the above results, for a replicator system determined by Eq. (*2a, b*) there exist two mutually excluding alternative possibilities. *First*, all replicators are becoming extinct (this possibility is determined by Eq. (*16*)). *Second*, there exists a molecular selection process which always leads to an unambiguous selection "decision" – to survival of the "best-fitted" *quasi-replicator*. Under the term quasi-replicator we mean such a set of replicators $X_1, X_2, \ldots, X_n$ in which the replicator X_p

is dominant and other ones are appearing in much smaller amount[1]. In biology this phenomenon corresponds to the notion *wild species*; a class of individuals which have fixed phenotypic properties in common, but the individual members of the species are slightly genotypically different. An optimally addapted phenotype is composed of individuals with scattered genotype narrowly around a defined avared sequence of nucleotide bases in DNA.

Selection Model with Two or More Substrates

In the previous section we have assumed that the studied selection process is realized for systems composed of only one substrate X_0. Now we remove[10] this constraint and let us consider a system composed of n replicators $X_1, X_2, \ldots, X_n$ and m different substrates $Y_1, Y_2, \ldots, Y_m$. The pattern of chemical reactions (A) and (B) is enlarged as follows

$$R_j\colon \ \square \xrightarrow{\varphi_j} Y_j\,, \tag{E}$$

$$R'_j\colon \ Y_j \xrightarrow{\psi_j y_j} \square\,, \tag{F}$$

$$A_{ij}\colon \ Y_j + X_i \xrightarrow{a_{ij}y_j x_i} X_i + X_i\,, \tag{G}$$

$$B_i\colon \ X_i \xrightarrow{b_i x_i} \square\,, \tag{H}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The third reaction (G) represents a replication of X_i with the participation of the substrate Y_j; the rate constant of this replication is a_{ij}. The assigned set of differential equations is

$$\dot{y}_j = \varphi_j - y_j(\psi_j + \sum_{i=1}^{n} a_{ij}x_i)\,, \quad (j = 1, 2, \ldots, m) \tag{30a}$$

$$\dot{x}_i = x_i(\sum_{j=1}^{m} a_{ij}y_j - b_i)\,, \qquad (i = 1, 2, \ldots, n) \tag{30b}$$

Its matrix form is

$$\dot{\boldsymbol{y}} = \boldsymbol{\varphi} - \mathrm{dg}\,(\boldsymbol{\psi} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{x})\,\boldsymbol{y}\,, \tag{31a}$$

$$\dot{\boldsymbol{x}} = \mathrm{dg}\,(\boldsymbol{A}\boldsymbol{y} - \boldsymbol{b})\,\boldsymbol{x}\,, \tag{31b}$$

where $\boldsymbol{A} = (a_{ij})$ is a matrix of rate constants from reaction (G). $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_m)^T$ and $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_m)^{\mathrm{T}}$ are column vectors of substrate inflows and substrate decaying rate constants, respectively. If the initial concentrations (at $t = 0$) of substrates and replicators are positive, then for each $t \geqq 0$ the substrate concentration vector $\boldsymbol{y} = \boldsymbol{y}(t) = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$ is positive and replicator concentration vector $\boldsymbol{x} = \boldsymbol{x}(t) = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is non-negative, and both are bounded from above,

$$0 < y_j(t) < \infty\,, \quad (j = 1, 2, \ldots, m) \tag{32a}$$

$$0 \leqq x_i(t) < \infty\,, \quad (i = 1, 2, \ldots, n)\,. \tag{32b}$$

Let I be a subset of the integer set $N = \{1, 2, \ldots, n\}$, we define with respect to this subset I a stationary state S_I as follows: All stationary substrate concentrations are positive (cf. Eq. (*32a*))

$$\bar{y}_j > 0\,, \quad (j = 1, 2, \ldots, m) \tag{33a}$$

and the stationary replicator concentrations are

$$i \in I \Rightarrow \bar{x}_i > 0\,, \tag{33b}$$

$$i \in I' \Rightarrow \bar{x}_i = 0\,,$$

where $I' = N \setminus I$ is composed of those integers of N that are not contained at I, i.e. $N = I \cup I'$ and $I \cap I' = \emptyset$. The positive entries of $\bar{\boldsymbol{x}}$ are collected at the column vector $\bar{\boldsymbol{x}}(I)$, its dimension is the same as the cardinality $s = |I|$ of the subset I. Similarly, if we omit in the matrix $\boldsymbol{A}$ the rows with indices not belonging to I, we get a submatrix $\boldsymbol{A}(I)$, this submatrix contains m columns and $s = |I|$ rows. The stationary states should be separately treated for $I = \emptyset$ (empty set, this stationary state is denoted by $S_\emptyset$) and for all other non-empty subsets $I \subset N$.

Stationary state $S_\emptyset$. The stationary concentration vectors are determined by

$$\bar{\boldsymbol{y}} = \mathrm{dg}^{-1}(\boldsymbol{\psi})\,\boldsymbol{\varphi}\,, \tag{34a}$$

$$\bar{\boldsymbol{x}} = \boldsymbol{0}\,, \tag{34b}$$

i.e. all replicator concentrations are vanishing.

Stationary state S_I. The set I is a non-empty subset of $N = \{1, 2, \ldots, n\}$, the stationary concentration vectors are determined by the following coupled matrix equations

$$\boldsymbol{A}^{\mathrm{T}}(I)\,\bar{\boldsymbol{x}}(I) = \mathrm{dg}^{-1}(\boldsymbol{y})\,\boldsymbol{\varphi} - \boldsymbol{\psi}\,, \tag{35a}$$

$$\boldsymbol{A}(I)\,\bar{\boldsymbol{y}} = \boldsymbol{b}(I)\,. \tag{35b}$$

We say that the stationary state S_I was properly selected if the above system has positive solution for $\bar{\boldsymbol{x}}(I)$ and $\bar{\boldsymbol{y}}$, in the opposite case we say that it does not exist and will be rejected from our forthcoming considerations.

Assuming that we know the stationary substrate concentration vector $\bar{\boldsymbol{y}}$, then the stationary replicator concentration vector $\bar{\boldsymbol{x}}(I)$ is determined by the linear problem

(*35a*). It has a solution[14] if and only if matrices $\boldsymbol{A}^{\mathrm{T}}(I)$ and $(\boldsymbol{A}^{\mathrm{T}}(I), \mathrm{dg}^{-1}(\boldsymbol{y})\,\boldsymbol{\varphi} - \boldsymbol{\psi})$ are of the same rank, i.e. $r[\boldsymbol{A}^{\mathrm{T}}(I)] = r[(\boldsymbol{A}^{\mathrm{T}}(I), \mathrm{dg}^{-1}(\boldsymbol{y})\,\boldsymbol{\varphi} - \boldsymbol{\psi})]$. This requirement can be fulfilled, in general, only for $|I| = s \leqq m$, i.e. for system with two or more substrates the maximal number of surviving replicators is equal to the number of substrates.

After this specification of stationary states we shall turn our attention to the construction of Jacobi matrices evaluated at these states. The Jacobi matrix of Eqs (*30a, b*) is

$$\boldsymbol{J} = \frac{\partial(\dot{\boldsymbol{y}}, \dot{\boldsymbol{x}})}{\partial(\boldsymbol{y}, \boldsymbol{x})} = \begin{pmatrix} -\mathrm{dg}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{\psi}) & -\mathrm{dg}(\boldsymbol{y})\,\boldsymbol{A}^{\mathrm{T}} \\ \mathrm{dg}(\boldsymbol{x})\,\boldsymbol{A} & \mathrm{dg}(\boldsymbol{A}\boldsymbol{y} - \boldsymbol{b}) \end{pmatrix}. \tag{36}$$

Its value at the stationary state S_0 is

$$\boldsymbol{J}(S_0) = \begin{pmatrix} -\mathrm{dg}(\boldsymbol{\psi}) & -\mathrm{dg}(\bar{\boldsymbol{y}})\,\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{0} & \mathrm{dg}(\boldsymbol{A}\bar{\boldsymbol{y}} - \boldsymbol{b}) \end{pmatrix}, \tag{37}$$

where the stationary substrate concentration vector $\bar{\boldsymbol{y}}$ is specified by Eq. (*34a*). Its eigenvalues are negative if

$$\boldsymbol{A}\bar{\boldsymbol{y}} < \boldsymbol{b}\,, \tag{38a}$$

where

$$\bar{\boldsymbol{y}} = \mathrm{dg}^{-1}(\boldsymbol{\psi})\,\boldsymbol{\varphi}\,. \tag{38a}$$

Hence, the stationary state S_0 is asymptotically stable if the condition (*38*) is satisfied

The Jacobi matrix (*36*) specified for S_I, where $I \neq \emptyset$, is of the following block structure

$$\boldsymbol{J}(S_I) = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{G} \\ \boldsymbol{0} & \boldsymbol{D} \end{pmatrix}, \tag{39a}$$

where

$$\boldsymbol{B} = \begin{pmatrix} -\mathrm{dg}^{-1}(\bar{\boldsymbol{y}})\,\mathrm{d}g(\boldsymbol{\varphi}) & -\mathrm{dg}(\bar{\boldsymbol{y}})\,\boldsymbol{A}^{\mathrm{T}}(I) \\ \mathrm{dg}(\bar{\boldsymbol{x}}(I))\,\boldsymbol{A}(I) & \boldsymbol{0} \end{pmatrix}, \tag{39b}$$

$$\boldsymbol{G} = -\mathrm{dg}(\bar{\boldsymbol{y}})\,\boldsymbol{A}^{\mathrm{T}}(I')\,, \tag{39c}$$

$$\boldsymbol{D} = \mathrm{dg}(\boldsymbol{A}(I')\,\bar{\boldsymbol{y}} - \boldsymbol{b}(I'))\,. \tag{39d}$$

The eigenvalues of $\boldsymbol{J}(S_I)$ are determined as the eigenvalues of either $\boldsymbol{B}$ or diagonal matrix $\boldsymbol{D}$. The eigenvalues originated by the diagonal matrix $\boldsymbol{D}$ are negative if

$$\boldsymbol{A}(I')\,\bar{\boldsymbol{y}} < \boldsymbol{b}\,. \tag{40}$$

The sign-problem of the eigenvalues originated by $\boldsymbol{B}$ will be treated as follows. Its eigenvalue problem looks like this

$$\boldsymbol{B}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{41}$$

where z_1 and z_2 are "subvectors" of an eigenvector of $\boldsymbol{B}$ asigned to the eigenvalue λ. Introducing Eq. (*39b*) into Eq. (*40*) we arrive at two coupled equations

$$-\mathrm{dg}^{-1}(\bar{y})\,\mathrm{dg}(\varphi)\,z_1 - \mathrm{dg}(\bar{y})\,\boldsymbol{A}^{\mathrm{T}}(I)\,z_2 = \lambda z_1\,, \tag{42a}$$

$$\mathrm{dg}(\bar{x}(I))\,\boldsymbol{A}(I)\,z_1 = \lambda z_2\,. \tag{42b}$$

Solving the first equation (*42a*) for the "subvector" z_1 and substituting this result in Eq. (*42b*) we get

$$\mathrm{dg}(\bar{x}(I))\,\boldsymbol{A}(I)\,[\lambda + \mathrm{dg}^{-1}(\bar{y})\,\mathrm{dg}(\varphi)]^{-1}\,\mathrm{dg}(\bar{y})\,\boldsymbol{A}^{\mathrm{T}}(I)\,z_2 = -\lambda z_2\,. \tag{43}$$

The obtained pseudoeigenvalue problem can be rewritten in

$$\boldsymbol{C}^{\mathrm{T}}(\lambda, I)\,\boldsymbol{C}(\lambda, I)\,t_2 = -\lambda t_2\,, \tag{44a}$$

where

$$t_2 = \mathrm{dg}^{-1,2}(\bar{x}(I))\,z_2\,, \tag{44b}$$

$$\boldsymbol{C}(I, \lambda) = \mathrm{dg}(\bar{y})\,[\lambda\,\mathrm{dg}(\bar{y}) + \mathrm{dg}(\varphi)]^{-1}\,\boldsymbol{A}^{\mathrm{T}}(I)\,\mathrm{dg}^{1,2}(\bar{x}(I))\,. \tag{44c}$$

Assuming that $r[\boldsymbol{A}(I)] = s$ (i.e. the rows of $\boldsymbol{A}(I)$ are linearly independent), then the matrix $\boldsymbol{C}^{\mathrm{T}}(\lambda, I)\,\boldsymbol{C}(\lambda, I)$ is manifestly positive definite, its eigenvalues should be positive, $-\lambda > 0$, or

$$\lambda < 0\,. \tag{45}$$

We have proved that the remaining eigenvalues of $\boldsymbol{J}(S_I)$, corresponding to the submatrix $\boldsymbol{B}$, are negative. It means that the stationary state S_I is asymptotically stable if the stationary substrate concentration vector $\bar{y}$ simultaneously satisfies the following two conditions

$$\boldsymbol{A}(I)\,\bar{y} = \boldsymbol{b}(I)\,, \tag{46a}$$

$$\boldsymbol{A}(I')\,\bar{y} < \boldsymbol{b}(I')\,. \tag{46b}$$

Hence the vector $\bar{y}$ should be taken from a convex set

$$\mathscr{K}(I) = \{y \in R_+^m;\ \boldsymbol{A}(I)\,\bar{y} = \boldsymbol{b}(I) \text{ and } \boldsymbol{A}(I')\,\bar{y} < \boldsymbol{b}(I')\}\,, \tag{47}$$

where R_+^m is the positive orthant of the vector space composed of m-dimensional column vectors.

Let us come back to Eqs (*35a, b*), they determine stationary vectors $\bar{y}$ and $\bar{x}(I)$ assigned to the state S_I. After simple algebra we rewrite these formulae in a form explicitly determinig the vectors $\bar{y}$ and $\bar{x}(I)$,

$$A(I)\,\mathrm{dg}^{-1}[A^{\mathrm{T}}(I)\,\bar{x}(I) + \psi]\,\varphi = b(I)\,, \tag{48a}$$

$$\bar{y} = \mathrm{dg}^{-1}[A^{\mathrm{T}}(I)\,\bar{x}(I) + \psi]\,\varphi\,. \tag{48b}$$

The non-linear expression (*48a*) determines the stationary replicator concentration vector $\bar{x}(I)$, the necessary condition that the solution $\bar{x}(I)$ exists is

$$A(I)\,\mathrm{dg}^{-1}(\psi)\,\varphi > b(I)\,. \tag{49}$$

This relation is in a contradiction with the necessary conditions (*38a, b*) for the stationary state S_0 to be asymptotically stable,

$$A\,\mathrm{dg}^{-1}(\psi)\,\varphi < b\,. \tag{50}$$

Among stationary states $S_\emptyset$ and S_I's (for $|I| \geqq 1$) is asymptotically stable either S_φ or a stationary state S_I with $|I| \geqq 1$.

Let us assume that for a preselected index subset I the stationary state S_I is asymptotically stable, then from Eqs (*48a, b*) and (*46b*) we get

$$A(I)\,\mathrm{dg}^{-1}[A^{\mathrm{T}}(I)\,\bar{x}(I) + \psi]\,\varphi = b(I)\,, \tag{51a}$$

$$A(I')\,\mathrm{dg}^{-1}[A^{\mathrm{T}}(I)\,\bar{x}(I) + \psi]\,\varphi < b(I')\,. \tag{51b}$$

The stationary concentration vector $\bar{x}(I)$ of S_I must simultaneously satisfy both relations (*51a, b*). We form new index subset $J = I \cup \{p\}$, where $p \notin I$ and $p \in \in \{1, 2, \ldots, n\}$, i.e. we have added to I and index p. In order to be the new formed state S_J asymptotically stable, the corresponding stationary concentration vector $\bar{x}(J)$ must satisfy an analogue of Eqs (*51a, b*)

$$A(J)\,\mathrm{dg}^{-1}[A^{\mathrm{T}}(J)\,\bar{x}(J) + \psi]\,\varphi = b(J)\,, \tag{52a}$$

$$A(J')\,\mathrm{dg}^{-1}[A^{\mathrm{T}}(J)\,\bar{x}(J) + \psi]\,\varphi < b(J')\,, \tag{52b}$$

where $J' = N \setminus J$. But, if we remmember, the index p was initially belonging to I', there is impossible that the above relations (*52a, b*) will be simultaneously satisfied, the state S_J either does not exist (the relation (*52a*) has not a solution $\bar{x}(J)$)or it is

unstable (the relation (*52b*) is not fulfilled). This means that two stationary states S_I and S_J, where $I \subset J$, could not be simultaneously asymptotically stable. The stationary states S_I and S_J may be simultaneously asymptotically stable only if their index subsets I and J are disjoint, i.e. $I \cap J = \emptyset$. The positive orthant of phase space is divided in two nonoverlapping domains of their attractivity. If the initial state is belonging to one of those domains, then all the trajectory will be situated at the given domain, i.e. its attractor – stationary state will win as $t \to \infty$.

Selection Model with External Inflows of Replicators

Recently, Krempaský and Kvetoň[17] have studied the interesting possibility how to obtain new qualitative features of the present model of selection when they introduced external inflows of replicators. We shall repeat their theoretical considerations in more general form, in particular the decay of substrate will be taken into account.

The pattern of chemical reactions of the present model looks like this

$$R_1\colon \square \xrightarrow{\varphi_0} \mathrm{X}_0\,, \tag{I}$$

$$R_2\colon \mathrm{X}_0 \xrightarrow{x_0\psi_0} \square\,, \tag{J}$$

$$R_i'\colon \square \xrightarrow{\varphi_i} \mathrm{X}_i\,, \tag{K}$$

$$A_i\colon \mathrm{X}_0 + \mathrm{X}_i \xrightarrow{a_i x_0 x_i} \mathrm{X}_i + \mathrm{X}_i\,, \tag{L}$$

$$B_i\colon \mathrm{X}_i \xrightarrow{b_i x_i} \square\,, \tag{M}$$

for $i = 1, 2, \ldots, n$. The third type of reactions (*K*) represents the above mentioned external inflows of replicators.

The corresponding system of differential equations is

$$\dot{x}_0 = \varphi_0 - x_0\Big(\psi_0 + \sum_{i=j}^{n} a_i x_i\Big)\,, \tag{53a}$$

$$\dot{x}_i = \varphi_i + x_i(a_i x_0 - b_i)\,, \quad (i = 1, 2, \ldots, n)\,. \tag{53b}$$

Its matrix form can be written as follows

$$\dot{x}_0 = \varphi_0 - x_0(\psi_0 + \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x})\,, \tag{54a}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{\varphi} + \mathrm{dg}(x_0\boldsymbol{a} - \boldsymbol{b})\,\boldsymbol{x}\,, \tag{54b}$$

where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)^{\mathrm{T}}$ is a column vector of replicator inflows, we shall

postulate that this vector is non-negative, i.e. none its entry is negative. Let J be an index subset of $N = \{1, 2, \ldots, n\}$, its elements determine the positive entries of the vector φ,

$$i \in J \Rightarrow \varphi_i > 0 \,, \tag{55a}$$

$$i \in J' \Rightarrow \varphi_i = 0 \,, \tag{55b}$$

where $J' = N \setminus J$. If the initial concentrations of substrate as well as replicators are positive, then for each $t \geqq 0$ these concentrations are bounded from above and

$$0 < x_0(t) < \infty \tag{56a}$$

$$0 < x_i(t) < \infty \,, \quad (i \in J) \tag{56b}$$

$$0 \leqq x_i(t) < \infty \,, \quad (i \in J') \tag{56c}$$

The stationary states of the system (*53a, b*) should be separately considered for two different possibilities in choising the index subset I. *First*, let us assume

$$I = J \,, \tag{57}$$

i.e. the positive stationary replicator concentrations $\bar{x}_i$ are those ones that have positive replicator inflows φ_i,

$$\bar{x}_i = \frac{\varphi_i}{b_i - a_i \bar{x}_0} \,, \quad (i \in J) \tag{58a}$$

$$\bar{x}_i = 0 \,, \qquad (i \in J') \tag{58b}$$

The stationary replicator concentration $\bar{x}_i$, for $i \in J$, is positive, this is ensured if

$$\bar{x}_0 < \min_{i \in J} \frac{b_i}{a_i} \,. \tag{59}$$

The stationary substrate concentration $\bar{x}_0$ is determined by Eq. (*53a*), we get that it should be the smallest positive solution of the following non-linear equation

$$\bar{x}_0 \left(\psi_0 + \sum_{i \in J} \frac{\varphi_i}{b_i/a_i - \bar{x}_0} \right) = \varphi_0 \,. \tag{60}$$

There is easy to see that it has the required solution $\bar{x}_0$ satisfying the constraint (*59*). This stationary state will be denoted by S_J.

Second, let I be an index subset obtained from the set when an additional index $p \in J'$ was added,

$$I = J \cup \{p\} . \tag{61}$$

Now, from Eq. (*53b*) specified for $i = p$ we get immediately the stationary substrate concentration

$$\bar{x}_0 = b_p/a_p . \tag{62}$$

The stationary replicator concentrations $\bar{x}_i$, for $i \neq p$, are (cf. Eq. (*58a*))

$$\bar{x}_i = \frac{\varphi_i}{b_i - a_i\bar{x}_0} , \quad (i \in J) \tag{63a}$$

$$\bar{x}_i = 0 . \qquad (i \in I') \tag{63b}$$

The stationary replicator concentration $\bar{x}_p$ will be determined by Eq. (*53a*),

$$\bar{x}_p = \frac{1}{a_p}\left(\frac{\varphi_0}{\bar{x}_0} - \psi_0 - \sum_{i \in J} a_i\bar{x}_i\right). \tag{64}$$

The positiveness of Eqs (*63a*) and (*64*) is achieved if the following inequalitise are fulfilled

$$\bar{x}_0 = b_p/a_p < \min_{i \in J}(b_i/a_i) , \tag{65a}$$

$$\varphi_0 > \frac{b_p}{a_p}\left(\psi_0 + \sum_{i \in J} \frac{\varphi_i}{b_i/a_i - b_p/a_p}\right). \tag{65b}$$

This stationary state will be denoted by S_I.

The Jacobi matrix of Eqs (*53a, b*)

$$\boldsymbol{J} = \frac{\partial(\dot{x}_0, \dot{\boldsymbol{x}})}{\partial(x_0, \boldsymbol{x})} = \begin{pmatrix} -\psi_0 - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} & -\boldsymbol{x}_0\boldsymbol{a}^{\mathrm{T}} \\ \mathrm{dg}(\boldsymbol{a})\,\boldsymbol{x} & \mathrm{dg}(x_0\boldsymbol{a} - \boldsymbol{b}) \end{pmatrix}. \tag{66}$$

Its values for the stationary states S_J and S_I are

$$\boldsymbol{J}(S_J) = \begin{pmatrix} -\varphi_0/\bar{x}_0 & -\bar{x}_0 a_j & -\bar{x}_0 a_j \\ a\bar{x}_i & \delta_{ij}(a_i\bar{x}_0 - b_i) & 0 \\ 0 & 0 & \delta_{ij}(a_i\bar{x}_0 - b_i) \end{pmatrix} \begin{matrix} i \in 0 , \\ i \in J , \\ i \in J' \end{matrix} \tag{67}$$

$$\begin{matrix} j = 0 & j \in J & j \in J' \end{matrix}$$

$$J(S_I) = \begin{pmatrix} -\varphi_0/\bar{x}_0 & -\bar{x}_0 a_p & -\bar{x}_0 a_j & -\bar{x}_0 a_j \\ a_p \bar{x}_p & 0 & 0 & 0 \\ a_i \bar{x}_i & 0 & \delta_{ij}(a_i \bar{x}_0 - b_i) & 0 \\ 0 & 0 & 0 & \delta_{ij}(a_i \bar{x}_0 - b_i) \\ j = 0 & j = p & p \in J & j \in I' \end{pmatrix} \begin{matrix} i = 0 \\ i = p \\ i \in J \\ i \in I' \end{matrix} \tag{68}$$

The Jacobi matrix $J(S_J)$ is of the triangle-block structure, therefore its eigenvalues are determined by the diagonal blocks. The right-down block $(i, j \in J')$ is diagonal matrix, therefore its diagonal entries should be negative,

$$\bar{x}_0 < \min_{i \in J'} (b_i/a_i) . \tag{69}$$

If we combine this inequality with Eq. (59) that ensures the positiveness of stationary replicator concentrations, we get

$$\bar{x}_0 < \min_{i \in N} (b_i/a_i) . \tag{70}$$

The left-up block (for $i, j \in J \cup \{0\}$) fo $J(S_J)$ has negative diagonal elements (cf. Eq. (59)). The eigenvalues of this block are determined as roots of the following non-linear equation

$$\sum_{i \in J} \frac{\bar{x}_0 \bar{x}_i a_i^2}{a_i \bar{x}_0 - b_i - \lambda} = \frac{\varphi_0}{\bar{x}_0} + \lambda . \tag{71}$$

Here the l.h.s. has singular points at negative values $\lambda = a_i \bar{x}_0 - b_i$. It is easy to show that its roots have always negative real parts, i.e. the eigenvalues of the left-up block of $J(S_J)$ have the negative real parts. This implies, the stationary state S_J is asymptotically stable if the conditiin (69) is fulfilled. Since the stationary concentration $\bar{x}_0$ is determined as the solution of non-linear equation (60), the condition (70) is not very useful to resolve whether the stationary state S_J is asymptotically stable or not. Let $p \in J'$ be an index selected in such a way that

$$b_p/a_p = \min_{i \in J'} (b_i/a_i) . \tag{72}$$

Due to the inequality (70) we have also

$$\bar{x}_0 < b_p/a_p . \tag{73}$$

Since the l.h.s. Eq. (60) is monotonously increasing function of $\bar{x}_0$ taken from the interval

$$0 \leqq \bar{x}_0 < \min_{i \in J} (b_i/a_i) , \tag{74}$$

then substituting $\bar{x}_0$ by the ratio b_p/a_p we get

$$\varphi_0 \leqq \frac{b_p}{a_p}\left(\psi_0 + \sum_{i\in J} \frac{\varphi_i}{b_i/a_i - b_p/a_p}\right). \tag{75}$$

The stationary state S_J is asymptotically stable if the substrate inflow φ_0 fulfils the above constraint, following Krempaský and Květoň[17] it is called the *subcritical regulation* of replicator system. In this case all replicators with positive external inflows are surviving, all others are becoming extinct.

The Jacobi matrix (68) has slightly more complicated block structure than (67). It has again block-triangle structure. The right-down block (for $i, j \in I'$) is diagonal, therefore its diagonal entries should be negative,

$$\bar{x}_0 < \min_{i\in I'} (b_i/a_i)\,. \tag{76}$$

Combining this result with Eq. (65a) we get

$$\bar{x}_0 < b_p/a_p = \min_{i\in N} (b_i/a_i)\,. \tag{77}$$

The eigenvalues of the left-up block (for $i, j \in J \cup \{0, p\}$) are determined as roots of

$$\frac{\bar{x}_0\bar{x}_p a_p^2}{-\lambda} + \sum_{i\in J} \frac{\bar{x}_0\bar{x}_i a_i^2}{a_i\bar{x}_0 - b_i - \lambda} = \frac{\varphi_0}{\bar{x}_0} + \lambda\,, \tag{78}$$

where the entries $a_i\bar{x}_0 - b_i$ are always negative for $i \in J$, cf. Eq. (65a). After simple algebraic considerations one can show that its roots have negative real parts. This gives that the stationary state S_I is asymptotically stable if the inequality (77) is satisfied.

Summarizing the above results we arrive at the following simple scheme how to distinguish different possibilities in the selection model with external inflow of replicators. At the beginning we determine the so-called "best-fitted" replicator X_p by

$$a_p/b_p = \max_{i\in I} (a_i/b_i)\,. \tag{79}$$

If $p \in J$ (i.e. the "best-fitted" replicator has also positive inflow), then S_J is the only asymptotically stable stationary state. In the opposite case, if $p \in J'$, the asymptotical stability of S_J or S_I (where $I = J \cup \{p\}$) depends on the values of substrate inflow φ_0. If we have the so-called *subcritical regulation*,

$$\varphi_0 \leqq \frac{b_p}{a_p}\left(\psi_0 + \sum_{i\in J} \frac{\varphi_i}{b_i a_i - b_p a_p}\right), \tag{80}$$

then S_J is asymptotically stable and S_I i sunstable. In the opposite case, if the inflow φ_0 satisfies the so-called *supercritical regulation* condition

$$\varphi_0 > \frac{b_p}{a_p}\left(\psi_0 + \sum_{i\in J}^{i} \frac{\varphi_i}{b_i a_i - b_p a_p}\right), \tag{81}$$

then the state S_I is asymptotically stable and S_J is unstable. Hence, we have very simple "tools" how to control in some extent replicator system by properly selected external substrate and replicator inflows.

CONCLUSION

We have studied three special kinetic models of selection processes with"constant-fluxes" constraints, originally postulated by Eigen[1] and elaborated in more deep form by Ebeling[9-10]. The theory was modified[13] by taking into account the possibility of substrate decaying, it removes a "catastrophic" state in which all replicators are becoming extinct and substrate concentration is linearly increased to infinity as $t \to \infty$. The first model involves incorrect reduplications of replicators called mutations, it permits to formulate very important concept of the quasireplicator. The second model covers the possibility that the system contains two or more substrates. These substrates are participating on the reduplication of replicators. We have shown that few replicators may survive while others are vanishing. The third model contains external constant and positiee replicator fluxes, then the system may be controlled by the fluxes. In our forthcoming communication we shall present an application of the theory to dynamical studies of greater kinetic patterns of replicators – hypercycles.

REFERENCES

1. Eigen M.: Naturwissenschaften *58*, 465 (1971).
2. Eigen M.: Quart. Rev. Biophys. *4*, 149 (1971).
3. Küppers B. O.: *Molecular Theory of Evolution. Outline of a Physico-Chemical Theory of the Origin of Life*. Springer, Berlin 1983.
4. Hofbauer J., Sigmund K.: *Evolutionstheorie und dynamische Systeme. Mathematische Aspekte der Selektion*. Paul Parey, Berlin 1984.
5. Schuster P., Sigmund K.: J. Theor. Biol. *100*, 533 (1983).
6. Jones B. L., Enns R. H., Rangekar S. S.: Bull. Math. Biol. *38*, 15 (9176).
7. Thompson C. J., McBridge J. L.: Math. Biosci. *21*, 127 (1974).
8. Tyson J. J. in: *Lectures on Mathematics in the Life Sciences* (S. A. Levine, Ed.). AMS Press, Rhode Island 1974.

9. Ebeling W.: *Strukturbildung bei Irreversiblen Procezzen.* Teubner, Leipzig 1976.
10. Ebeling W., Feistel R.: *Physik der Selbstorganisation und Evolution.* Akademie Verlag, Berlin 1982.
11. Hirsch M. W., Smale S.: *Differential Equations, Dynamical System, and Linear Algebra.* Academic Press, New York 1975.
12. Clarke B. L.: Adv. Chem. Phys. *43*, 1 (1980).
13. Kvasnička V.: Chem. Zvesti, submitted.
14. Lancaster P.: *Theory of Matrices.* Academic Press, New York 1969.
15. Volpert A. I.: Mat. Sbor. *88*, 578 (1972).
16. Volpert A. I., Chudyaev C. I.: Analiz v klassakh razryvnykh funktsii i uravneniya matematicheskoi fiziki. Nauka, Moscow 1975.
17. Krempaský J., Květoň R.: Gen. Physiol. Biophys. *5*, 517 (1986).

Translated by the author.